Math 1B – Calculus – A few divergent integrals worked through – Spring '10

1. Explain why the Integral $\int_0^3 \frac{1}{(x-1)^2(x-2)^2}\,dx$ is improper and then evaluate it.

SOLN: There are vertical asymptotes at $x = 1$ and $x = 2$ which are interior to the interval of integration.

$$\int_0^3 \frac{1}{(x-1)^2(x-2)^2}\,dx = \int_0^{3/2} \frac{1}{(x-1)^2(x-2)^2}\,dx + \int_{3/2}^3 \frac{1}{(x-1)^2(x-2)^2}\,dx$$

To simplify matters, use symmetry and substitute $u = x - 3/2$ whence the integral becomes

$$\int_{-3/2}^{3/2} \frac{1}{(u+1/2)^2(x-1/2)^2}\,dx = \int_{-3/2}^{0} \frac{1}{(u+1/2)^2(x-1/2)^2}\,dx + \int_0^{3/2} \frac{1}{(u+1/2)^2(x-1/2)^2}\,dx$$

and then substitute $v = -u$ into the first one so that it is identical to the second and

$$\int_{-3/2}^{3/2} \frac{1}{(u+1/2)^2(x-1/2)^2}\,dx = 2\int_0^{3/2} \frac{1}{(u+1/2)^2(x-1/2)^2}\,dx$$

$$= 2\int_0^{3/2} \frac{2}{u+1/2} + \frac{1}{(u+1/2)^2} - \frac{2}{x-1/2} + \frac{1}{(x-1/2)^2}\,dx$$

$$= 4\ln(4) + 3 - 4\lim_{b\to 1/2^-}\int_0^b \frac{dx}{x-1/2} + 4\lim_{b\to 1/2^+}\int_{3/2}^b \frac{dx}{x-1/2}$$

$$+2\lim_{b\to 1/2^-}\int_0^b \frac{dx}{(x-1/2)^2} + 2\lim_{b\to 1/2^+}\int_b^{3/2} \frac{dx}{(x-1/2)^2}$$

$$= 4\ln(4) + 3 - 4\lim_{b\to 1/2^-}\ln\left|b-\frac{1}{2}\right| + 4\ln\left|\frac{1}{2}\right| + 4\lim_{b\to 1/2^+}\ln\left|b-\frac{1}{2}\right|$$

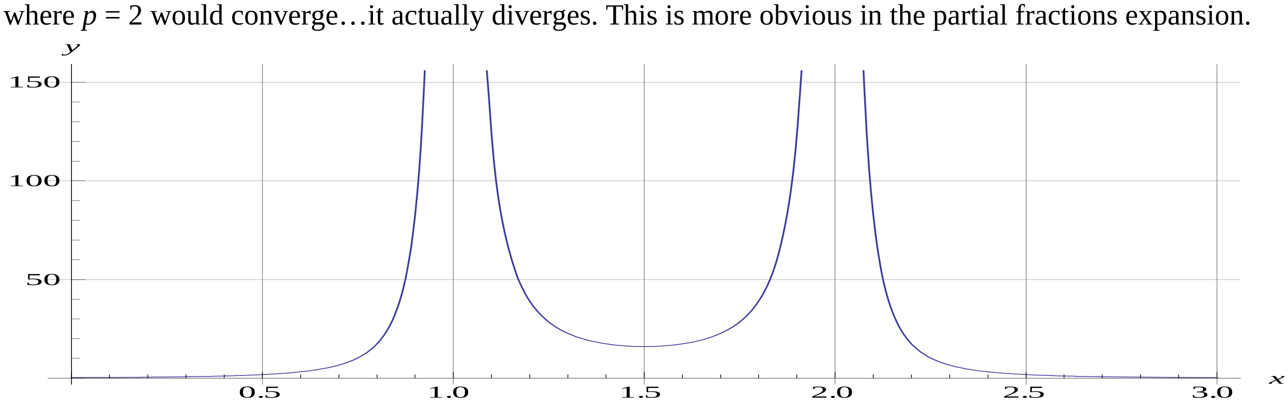
$$-\lim_{b\to 1/2^-}\frac{2}{b-1/2} - 4 + \lim_{b\to 1/2^+}\frac{2}{b-1/2} - 2$$

$$= 4\ln(2) + 3 - 4\lim_{\varepsilon\to 0}\ln|-\varepsilon| + 4\lim_{\varepsilon\to 0}\ln|\varepsilon|$$

$$\infty - 4 + \infty - 2$$

$$= 4\lim_{\varepsilon\to 0}\ln|1| + \infty = \infty$$

So, while your first intuition about this integral may be that it's going to go like $1/x^p$ around a vertical asymptote where $p = 2$ would converge…it actually diverges. This is more obvious in the partial fractions expansion.



2. Explain why the Integral $\int_0^3 \frac{1}{\sqrt{|x-1|}\sqrt{|x-2|}}\,dx$ is improper and then evaluate it.

SOLN: There are vertical asymptotes at x = 1 and x =2 which are interior to the interval of integration. To simplify matters, use symmetry and substitute $u = x - 3/2$ whence the integral becomes

$$\int_0^3 \frac{1}{\sqrt{x-1}\sqrt{x-2}}\,dx = \int_{-3/2}^{3/2} \frac{1}{\sqrt{u-1/2}\sqrt{u+1/2}}\,du = \int_{-3/2}^{3/2} \frac{1}{\sqrt{u^2-1/4}}\,du = \int_{-3/2}^{3/2} \frac{2}{\sqrt{4u^2-1}}\,du$$

Now use y-axis symmetry and then split the absolute value into two pieces:

$$\int_{-3/2}^{3/2} \frac{2}{\sqrt{|4u^2-1|}}\,du = 2\int_0^{3/2} \frac{2}{\sqrt{|4u^2-1|}}\,du = 2\int_0^{1/2} \frac{2}{\sqrt{1-4u^2}}\,du + 2\int_{1/2}^{3/2} \frac{2}{\sqrt{4u^2-1}}\,du$$

$$= \int_0^{\pi/2} \frac{\cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}} + \int_{\pi/2}^{\cos^{-1}1/3} \frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta-1}}\,d\theta$$

which involve the trig substitutions: $2u = \sin\theta \Rightarrow 2du = \cos\theta\,d\theta$ and $2u = \sec\theta \Rightarrow 2du = \sec\theta\tan\theta\,d\theta$

$$\int_0^{\pi/2} d\theta + \int_{\pi/2}^{\cos^{-1}1/3} \sec\theta\,d\theta = \frac{\pi}{2} + \lim_{b\to\pi/2^-} \ln|\sec\theta+\tan\theta|\Big|_b^{\cos^{-1}1/3}$$

$$= \frac{\pi}{2} + \ln\left|3+\frac{\sqrt{8}}{3}\right| - \lim_{b\to\pi/2^-} \ln|\sec b+\tan b|$$

$$= \frac{\pi}{2} + \ln\left|3+\frac{\sqrt{8}}{3}\right| + \infty = \infty$$